THE CLASSIFICATION OF STUNTED PROJECTIVE SPACES BY STABLE HOMOTOPY TYPE

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ABSTRACT. A complete classification of stable homotopy types of complex and quaternionic stunted projective spaces, denoted by CP_n^k and QP_n^k respectively, is obtained. The necessary conditions for such equivalences are found using K-theory and various characteristic classes introduced originally by J. F. Adams. As a by-product one finds the J-orders of the Hopf bundles over CP^n and QP^n respectively. The algebraic part is rather involved. Finally a homotopy theoretical argument yields the constructions of such homotopy equivalences as are allowed by the fulfillment of the necessary conditions.

1. **Introduction.** The results of this paper were announced in [9].

Let FP^n be the projective space of dimension n over F, where F is either the field of complex numbers $\mathbf{C}$ or the quaternions $\mathbf{Q}$. We have the natural cofibrations

$$FP^{m-1} \to FP^n \to FP_m^n$$

for $m \leqslant n$.

Let ξ denote the Hopf bundle over FP^n. It is well known that the stunted projective space FP_k^{n+k} can be identified with the Thom space of $k\xi$ over FP^n which we denote by $(FP^n)^{k\xi}$.

In this paper we study the stable homotopy type of stunted projective spaces FP_k^{n+k} for $F = \mathbf{C}$ and $F = \mathbf{Q}$ and obtain a complete classification. This classification is given in terms of the J-orders of the Hopf bundles. We denote by A_n the J-order of the Hopf bundle over $\mathbf{C}P^n$ and by B_n the J-order of the quaternion Hopf bundle over $\mathbf{Q}P^n$. With this notation the classification can be stated as follows:

THEOREM 1.1. *The spaces* $\mathbf{C}P_k^{n+k}$ *and* $\mathbf{C}P_l^{n+l}$ *are of the same stable homotopy type if and only if one of the following conditions hold* ($n \neq 2, 4$):

(i) $k - l \equiv O(A_n)$,

(ii) $k - l \equiv O(A_{n-1})$ *and* $k + l \equiv O(A_n)$,

(iii) $k - l \equiv O(A_{n-1})$ and $k + l + 2(n + 1) \equiv O(A_n)$.

THEOREM 1.2. *The spaces* $\mathbf{Q}P_k^{n+k}$ *and* $\mathbf{Q}P_l^{n+l}$ *are of the same stable homotopy type if and only if one of the following conditions hold*:
(i) $k - l \equiv O(B_n)$,
(ii) $k - l \equiv O(B_{n-1})$ *and* $k + l \equiv O(B_n)$.

The apparent difference between the two statements is due to the fact that the family of stunted complex projective spaces is closed under S-duality while the family of stunted quaternionic projective spaces is not. Note that conditions (ii) and (iii) in Theorem 1.1 correspond under S-duality.

In [6] we have proven that the conditions in Theorem 1.1 are necessary and have completed the classification for n odd by observing that $A_n = A_{n-1}$ for that case. This paper thus completes and extends [6]. We also give a proof of the result of F. Sigrist and U. Sutter [15] on the number B_n.

Let $\nu_p(m)$ be the largest integer for which $p^{\nu_p(m)}$ divides m. The result is

THEOREM 3.4. *The J-order* B_n *of the Hopf bundle over* $\mathbf{Q}P^n$ *is given by*

$$B_n = \prod p^{\nu_p(B_n)}$$

where

$$\nu_2(B_n) = \max\{2n + 1, 2j + \nu_2(j) | 1 \leqslant j \leqslant n\}$$

and

$$\nu_p(B_n) = \max\{j + \nu_p(j) | 1 \leqslant j \leqslant 2n/(p - 1)\}$$

when p is an odd prime.

In [6] we used the fact, proven by Adams and Walker [3], that $A_n = M_{n+1}$, the Atiyah-Todd number. In this paper we shall also give a short proof of that fact.

The classification is accomplished in two steps. First, similar to what was done in [6], certain necessary conditions are obtained. This is achieved by using certain characteristic classes and the following simple proposition:

PROPOSITION 1.4. *If* $f: X \to Y$ *is a homotopy equivalence then the diagram*

$$
\begin{array}{ccc}
\widetilde{KF}(Y) & \xrightarrow{\;f^!\;} & \widetilde{KF}(X) \\
\downarrow{ch} & & \downarrow{ch} \\
\widetilde{H}^*(Y: Q) & \xrightarrow{\;f^*\;} & \widetilde{H}^*(X, Q)
\end{array}
$$

is commutative and both f^* *and* $f^!$ *are invertible transformations.*

Here $\widetilde{KF}$ may stand for real ($F = O$), complex ($F = U$), or sympletic ($F = \text{Sp}$) reduced K-theory and the Chern character is interpreted in the appropriate way.

Once the necessary conditions are obtained their sufficiency is established by constructing the desired homotopy equivalences. The low-dimensional cases present some anomalies and must be treated separately.

As corollaries we also obtain the representations

$$\mathcal{E}(FP_n^m) \to \text{Aut } (H^*(FP_n^m, Z))$$

where $\mathcal{E}(X)$ is the group of stable homotopy equivalences of X and $F = \mathbf{C}$ or $\mathbf{Q}$.

2. Characteristic classes. Let ξ be a vector bundle over X. We say that ξ is KF-orientable if there exists an isomorphism $KF(X) \xrightarrow{\phi} \widetilde{KF}(X^\xi)$. A choice of one such isomorphism is a KF-orientation. To simplify our exposition we shall say, unless otherwise stated, that ξ is KO-orientable if ξ has a Spin($8n$)-structure and that ξ is KU-orientable if ξ has a $U(n)$-structure.

Let U^H and U^F denote the Thom classes for integral cohomology H^* and $\widetilde{KF}$-theory respectively. Adams in [2] defines a class $bh(\xi) \in H^*(X; Q)$ by

$$ch U^F = U^H \cup bh(\xi)$$

where the character is taken after complexification if KO is the theory in question. Adams also defines classes $\rho^k(\xi)$ (these are due to Bott [5]) by

$$\psi^k U^F = U^F \rho^k(\xi).$$

If ξ and η are vector bundles which are J-equivalent then their classes bh and ρ^k satisfy

(2.1) $$bh(\xi) = bh(\eta)ch(1 + x),$$

(2.2) $$\rho^k(\xi) \stackrel{k}{=} \rho^k(\eta)\psi^k(1 + x)/(1 + x)$$

for some element $x \in \widetilde{KF}(X)$, and all k. The equality (2.2) denoted by $=^k$ means that the two sides may differ by a factor k^r for some r.

The solution of the Adams conjecture by Adams [1] and Quillen [13] establishes (2.2) as a necessary and sufficient condition for two *real* vector bundles to be J-equivalent.

In special cases conditions (2.1) and (2.2) are equivalent. Let $F = O$ or U.

PROPOSITION 2.3. *Suppose $KF(X)$ has no torsion. If ξ_1 and ξ_2 are KF-orientable vector bundles and if there exists a class $x \in KF(X)$ such that $bh(\xi_1) = bh(\xi_2)ch(1 + x)$, then for all integers $k \neq 1$, we have: $\rho^k(\xi_1) =^k \rho^k(\xi_2)\psi^k(1 + x)/(1 + x)$.*

PROOF. Recall the definition of $\bar{\psi}^k : H^{\mathrm{ev}}(X; Q) \to H^{\mathrm{ev}}(X; Q)$ by $\bar{\psi}^k(u)$ $= k^{1/2 \dim u} u$. Then because of the hypothesis, we have $ch\psi^k = \bar{\psi}^k ch$.

Let ξ be a KF-orientable bundle of real dimension $2n$. Then

$$ch\psi^k U_\xi^F = \bar{\psi}^k ch U_\xi^F = \bar{\psi}^k (U_\xi^H \cdot bh(\xi)) = k^n U_\xi^H \bar{\psi}^k (bh(\xi)).$$

Also

$$ch\psi^k U_\xi^F = ch(U_\xi^F \cdot \rho^k(\xi)) = U_\xi^H \cdot bh(\xi) ch\rho^k(\xi).$$

Hence

$$k^n \bar{\psi}^k (bh(\xi)) = bh(\xi) ch\rho^k(\xi).$$

Now consider KF-orientable bundles ξ_1 and ξ_2 of dimensions $2n_1$ and $2n_2$ respectively. We have

(i)
$$bh(\xi_1) ch\rho^k(\xi_1) = bh(\xi_2) ch[(1 + x)\rho^k(\xi_1)]$$

and

(ii)
$$k^{n_1} \bar{\psi}^k (bh(\xi_1)) = k^{n_1} \bar{\psi}^k (bh(\xi_2) ch(1 + x))$$
$$= k^{n_1 - n_2} bh(\xi_2) ch\rho^k(\xi_2) \bar{\psi}^k ch(1 + x).$$

By the above, (i) and (ii) are equal. Now concentrating on the right-hand sides of (i) and (ii), we have

$$bh(\xi_2) ch[(1 + x)\rho^k(\xi_1) - k^{n_1 - n_2}\rho^k(\xi_2)\psi^k(1 + x)] = 0.$$

But $bh(\xi_2)chy = 0$ if and only if $y = 0$, because, $ch(U_{\xi_2}^F \cdot y) = U_{\xi_2}^H$ $\cdot bh(\xi_2)chy$ and ch is a monomorphism. Thus

$$(1 + x)\rho^k(\xi_1) - k^{n_1 - n_2}\rho^k(\xi_2)\psi^k(1 + x) = 0$$

and the proposition follows.

Note that the condition that $KF(X)$ be torsion free in (2.3) is not necessary. It would suffice to ask that under some canonical splitting (e.g. filtration) $\rho^k(\xi_1)$ lie in the part of $KF(X)$, on which ch is a monomorphism.

When dealing with spaces whose KF-theory is torsion free it is convenient to extend the character monomorphism $ch: KF(X) \to H^*(X; Q)$ to $KF(X)$ $\otimes Q$. The resulting homomorphism is an isomorphism onto $H^{\mathrm{ev}}(X; Q)$ when $F = U$ and onto $\sum_i H^{4i}(X; Q)$ when $F = O$. The inverse of this isomorphism will be denoted by ch^{-1}.

We introduce now a new characteristic class $Bh(\xi) \in KF(X) \otimes Q$ as follows:

$$(2.4) \qquad ch^{-1} U^H = U^F \cdot BH(\xi)^{-1}.$$

This definition makes sense since $Bh(\xi)$ is an element in the multiplicative group $1 + (\widetilde{KF}(X) \otimes Q)$. Note that Bh (just as ρ^k) is multiplicative, i.e. $Bh(\xi + \eta) = Bh(\xi)Bh(\eta)$ and since some multiple of any F-bundle is KF-orientable we can extend Bh over all of $KF(X)$ defining $Bh(\xi) = [Bh(m\xi)]^{1/m}$. Applying the character to both sides of the defining relation for $Bh(\xi)$ we find that

$$(2.5) \qquad Bh(\xi) = ch^{-1} bh(\xi).$$

The following are useful corollaries of Proposition 2.3.

COROLLARY 2.6. *If $KO(X)$ is torsion free then the KO-oriented bundles ξ_1 and ξ_2 are J-equivalent if and only if there exists $x \in \widetilde{KO}(X)$ such that $bh(\xi_1) = bh(\xi_2)ch(1 + x)$, i.e. $Bh(\xi_1) = (1 + x)Bh(\xi_2)$.*

In particular, if ξ_2 is the trivial bundle then $bh(\xi_2) = 1$ and we obtain

COROLLARY 2.7. *If $KO(X)$ is torsion free then a bundle ξ is J-trivial if and only if there exists $x \in \widetilde{KO}(X)$ such that $bh(\xi) = ch(1 + x)$, i.e. $Bh(\xi) = (1 + x)$— an element in $1 + \widetilde{KO}(X)$.*

Another formulation of (2.7) is

COROLLARY 2.8. *If $KO(X)$ is torsion free, then a bundle ξ is J-trivial if and only if U_ξ^H is in the image of the Chern character.*

We should point out that the "integrality" of $Bh(\xi)$ will not be used directly to compute the J-order of the Hopf bundle. The usefulness of the condition on $Bh(\xi)$ will become apparent in the classification of stunted projective spaces.

We now turn our attention to the cannibalistic classes ρ^k. Let ξ be a KF-orientable vector bundle of real dimension $2n$. Consider the Chern character of $\rho^k(\xi)$.

$$ch\psi^k U^F = \overline{\psi}^k ch U^F = \overline{\psi}^k (U^H bh(\xi))$$

$$= \overline{\psi}^k U^H \cdot \overline{\psi}^k bh(\xi) = k^n U^H \overline{\psi}^k bh(\xi).$$

Since $ch(U^F \rho^k(\xi)) = U^H bh(\xi) ch\rho^k(\xi)$, we obtain

$$ch\rho^k(\xi) = k^n \overline{\psi}^k bh(\xi)/bh(\xi).$$

In particular, if $KF(X)$ is torsion free then

$$(2.9) \qquad \rho^k(\xi) = k^n \psi^k Bh(\xi)/Bh(\xi).$$

COROLLARY 2.10. *If ξ is a real oriented vector bundle of dimension $2n$ such that for some $k \neq 1$ we have*

$$\rho^k(\xi) = k^n \psi^k(1 + x)/(1 + x)$$

for $x \in \widetilde{KO}(X)$ then $bh(\xi) = ch(1 + x)$. In particular, if $KO(X)$ is torsion free, the bundle ξ is J-trivial.

PROOF. Applying the Chern character to both sides of the equality we obtain

$$\bar{\psi}^k bh(\xi)/bh(\xi) = \bar{\psi}^k ch(1 + x)/ch(1 + x)$$

and the corollary follows from

LEMMA 2.11. *The transformation $z \to \bar{\psi}^k z/z$ on the group $1 + \bar{H}^{ev}(X; Q)$ is an automorphism for all $k \neq 1$.*

PROOF. The transformation in question is clearly a group homomorphism. Let $z = 1 + \sum_{i>0} x_i$ with $x_i \in H^{2i}(X; Q)$. Then

$$\bar{\psi}^k z = 1 + \sum k^i x_i$$

and if $\bar{\psi}^k z/z = 1$ then $1 + \sum k^i x_i = 1 + \sum x_i$ and because $k \neq 1$, we obtain $x_i = 0$ for all i. The proof that $z \to \bar{\psi}^k z/z$ is an epimorphism is just as simple.

Note that the "conjugate" homomorphism $x \to \psi^k x/x$ on $1 + \widetilde{KF}(X)$ is also an automorphism for $k \neq 1$ whenever $KF(X)$ is torsion free. ⌡

PROPOSITION 2.12. *Let $KF(X)$ be torsion free. Then $z \in \widetilde{KF}(X) \otimes Q$ is integral if and only if $\psi^p(1 + z)/(1 + z)$ is integral, for every prime p.*

PROOF. It suffices to prove that if $z \notin \widetilde{KF}(X)$ and $pz \in \widetilde{KF}(X)$ then $\psi^p(1 + z)/(1 + z)$ is not integral. We can also assume that $z + x$, where $x \in \widetilde{KF}(X) \otimes Q$ is of higher filtration than z, is not integral for any such x.

First consider the case $F = U$. Since $z \in \widetilde{KU}(X) \otimes Q$ it has filtration $k \geq 1$. Now ψ^p preserves filtration and $\psi^p z = p^k z + a$ where a is an element of higher filtration. We have

$$\psi^p(1 + z)/(1 + z) = (1 + p^k z + a)(1 - z + z^2 - \cdots)$$

$$= 1 + (p^k - 1)z + \text{elts of higher filtration}$$

and this shows that $\psi^p(1 + z)/(1 + z)$ is not integral, as was claimed. The case $F = O$ follows by the same argument since when $KO(X)$ is torsion free the formula for $\psi^p z$ still holds with $k = $ (filtration of cz) ≥ 1.

3. **The J-order of the Hopf bundles.** Let $\mathbf{Q}P^\infty$ and $\mathbf{Q}P^n$ denote the infinite- and n-dimensional projective spaces over the quaternions. Let ξ denote the Hopf bundle over $\mathbf{Q}P^\infty$ as well as its restrictions to $\mathbf{Q}P^n \subset \mathbf{Q}P^\infty$.

Recall (e.g. [14]) that $KU(\mathbf{Q}P^n) = Z[\mu]/(\mu^{n+1})$ where $\mu = \xi - 2$ and c: $KO(\mathbf{Q}P^n) \to KU(\mathbf{Q}P^n)$ is a monomorphism whose image is the subring generated by 2μ and μ^2. The complexification of the real vector bundle ξ is $2\mu + 4$. The ring $KU(\mathbf{Q}P^\infty)$ can be thought of as the ring of power series in μ, and $KO(\mathbf{Q}P^\infty)$ as the corresponding subring generated by 2μ and μ^2.

We choose $y \in H^4(\mathbf{Q}P^\infty, Z)$ so that $1, y, y^2, \ldots$ gives a basis for $H^*(\mathbf{Q}P^\infty)$ and $H^*(\mathbf{Q}P^n)$ in such a way that $y \in H^4(\mathbf{Q}P^n)$ is the characteristic class of the normal bundle of the natural inclusion $\mathbf{Q}P^n \subset \mathbf{Q}P^{n+1}$ i.e. the Hopf bundle ξ over $\mathbf{Q}P^n$. We denote by y the fundamental class in $H^*(\mathbf{Q}P^\infty)$ as well as its restrictions to $H^*(\mathbf{Q}P^n)$.

We have short exact sequences ($k \leqslant n$)

$$0 \to H^*(\mathbf{Q}P_k^n) \xrightarrow{j^*} H^*(\mathbf{Q}P^n) \xrightarrow{i^*} H^*(\mathbf{Q}P^{k-1}) \to 0$$

and

$$0 \to KU(\mathbf{Q}P_k^n) \xrightarrow{j^!} KU(\mathbf{Q}P^n) \xrightarrow{i^!} KU(\mathbf{Q}P^{k-1}) \to 0$$

corresponding to the cofibration

$$\mathbf{Q}P^{k-1} \to \mathbf{Q}P^n \to \mathbf{Q}P_k^n.$$

Identifying $\tilde{H}^*(\mathbf{Q}P_k^n)$ with $\ker j^*$ we obtain the basis $y^k, y^{k+1}, \ldots,$ y^n for $\tilde{H}^*(\mathbf{Q}P_k^n)$ and similarly $\mu^k, \ldots, \mu^n$ as basis for $\widetilde{KU}(\mathbf{Q}P_k^n)$. Note that $\Lambda_t \mu^t$ with $\Lambda_t = 1$ if t is even and $\Lambda_t = 2$ if t is odd give a basis for $cKO(\mathbf{Q}P_k^n)$.

We also have the identification $\mathbf{Q}P_k^{m+k} \simeq (\mathbf{Q}P^m)^{k\xi}$ and a basis for $\tilde{H}^*(\mathbf{Q}P_k^{m+k})$ via the Thom isomorphism. One easily checks that, with our choice of y, these two bases coincide. The same holds true for $KU(\mathbf{Q}P_k^{m+k})$. Also for $KO(\mathbf{Q}P_k^{m+k})$ whenever k is even.

Note that $U_{k\xi}^H = y^k$, $U_{k\xi}^U = \mu^k = cU_{k\xi}^O$ for k even. Since

$$c^{-1}ch^{-1}U_{k\xi}^H = U_{k\xi}^O \cdot Bh_O(-k\xi) \quad \text{and} \quad ch^{-1}U_{k\xi}^H = U_{k\xi}^U \cdot Bh_U(-k\xi),$$

we see that $cBh_O(-k\xi) = Bh_U(-k\xi)$.

In what follows we shall identify $\widetilde{KO}(\mathbf{Q}P^n)$ with its complexification and consider it a subring of $\widetilde{KU}(\mathbf{Q}P^n)$. Note that

$$KO(\mathbf{Q}P^n) \otimes Q \xrightarrow{c \otimes 1} KU(\mathbf{Q}P^n) \otimes Q$$

is an isomorphism. With this convention, $Bh_O(\xi) = Bh_U(\xi)$, and we will denote it simply by $Bh(\xi)$.

PROPOSITION 3.1. *In* $\mathbf{Q}P^n$ *the following statements are equivalent.*
(i) *The bundle* $k\xi$ *is J-trivial.*
(ii) *The class* $Bh(k\xi)$ *belongs to* $1 + \widetilde{KO}(\mathbf{Q}P^n)$.

(iii) *The class $\psi^p Bh(k\xi)/Bh(k\xi)$ belongs to $1 + \widetilde{KO}(\mathbf{Q}P^n)$ for all primes p.*
(iv) *The class $(\psi^p \mu/p^2 \mu)^k$ belongs to $1 + \widetilde{KO}(\mathbf{Q}P^n)$ for all primes p.*

PROOF. Proposition (2.5) shows that (i) and (ii) are equivalent. Proposition (2.12) shows that (ii) and (iii) are equivalent. Now from (2.9), we have

(1)
$$\rho^p(2\xi) = p^4 \psi^p Bh(2\xi)/Bh(2\xi).$$

Also

$$\psi^p U_{2\xi}^0 = U_{2\xi}^0 \cdot \rho^p(2\xi),$$

hence $\rho^p(2\xi) = \psi^p U_{2\xi}^0/U_{2\xi}^0$ and identifying $\rho^p(2\xi)$ with its image in $KU(\mathbf{Q}P^n)$, we have

(2)
$$\rho^p(2\xi) = (\psi^p \mu/\mu)^2.$$

From (1) and (2) we obtain

$$\psi^p Bh(2\xi)/Bh(2\xi) = (\psi^p \mu/p^2 \mu)^2$$

and hence (note that k must be even)

$$\psi^p Bh(k\xi)/Bh(k\xi) = (\psi^p \mu/p^2 \mu)^k;$$

thus (iii) and (iv) are equivalent and (3.1) follows.

In [7] we gave the following explicit formula for ψ^p in $\mathbf{Q}P^\infty$:

(3.2)
$$\psi^p(\mu) = \sum_{m=1}^{p} \binom{p+m-1}{2m-1} \frac{p}{m} \mu^m.$$

We now will show the following:

PROPOSITION 3.3. *Let $\mu = \xi - 2$ denote the generator in $KU(\mathbf{Q}P^\infty)$. Then $\psi^2 \mu/\mu = 4 + \mu$ and for p odd*

$$\psi^p(\mu)/\mu = (p\lambda + \mu^{(p-1)/2})^2$$

where $\lambda \in 1 + \widetilde{KU}(\mathbf{Q}P^\infty)$.

PROOF. We begin by showing that $\psi^p(\mu)/\mu$ is a square whenever p is odd. Consider the standard fibration $\pi: \mathbf{C}P^\infty \to \mathbf{Q}P^\infty$. Then $\pi^* \mu = \xi + \xi^{-1} - 2$ where ξ is the Hopf bundle over $\mathbf{C}P^\infty$ and

$$\psi^p \pi^*(\mu) = \xi^p + \xi^{-p} - 2 = (\xi^p - 1)^2/\xi^p.$$

Since π^* is a monomorphism we find $\psi^p(\mu)$ by expressing $x^p + x^{-p} - 2$ as a polynomial in $x + x^{-1} - 2$ and setting $\mu = x + x^{-1} - 2$. Now $\alpha + \alpha^{-1} - 2$ is

a root of that polynomial, whenever α is a root of $x^p - 1$. The roots of $x^p - 1$ are $e^{2\pi i s/p}$ for $0 \leqslant s \leqslant p - 1$ and so the roots of $\psi^p(\mu)$ as a polynomial in μ are $e^{2\pi i s/p} + e^{2\pi i(p-s)/p} - 2$. If p is odd, one of the roots ($s = 0$) is zero and all others are double roots. Thus $\psi^p(\mu) = \mu[R(\mu)]^2$ and $R(\mu)$ is an integral polynomial since $\psi^p(\mu)/\mu$ is. It follows from (3.2) that the coefficient of μ^m in $[R(\mu)]^2$ for $m < (p + 1)/2$ is divisible by p^2 and so $R(\mu) = p\lambda + \mu^{(p-1)/2}$ as was claimed. The formula for $\psi^2(\mu)/\mu$ is also contained in (3.2).

We are now ready to prove:

THEOREM 3.4. *The J-order B_n of the Hopf bundle over $\mathbf{Q}P^n$ is given by*

$$B_n = \prod p^{\nu_p(B_n)}$$

where

$$\nu_2(B_n) = \max \{2n + 1, 2j + \nu_2(j) | 1 \leqslant j \leqslant n\}$$

and

$$\nu_p(B_n) = \max \{j + \nu_p(j) | 1 \leqslant j \leqslant 2n/(p - 1)\}$$

when p is an odd prime.

PROOF. By Stiefel-Whitney class consideration B_n is even, so that we may apply (3.1). Therefore it suffices to prove that B_n is the least integer s for which

(3.5) $$(\psi^p \mu/p^2 \mu)^s \in 1 + \widetilde{KO}(\mathbf{Q}P^n),$$

for all primes p.

We need a lemma on divisibility of binomial coefficients.

LEMMA 3.6. *Let p be a prime and suppose p^j divides $\binom{s}{j}$ for all $1 \leqslant j \leqslant q$. Then $\nu_p(s) - \nu_p(j) = \nu_p\binom{s}{j}$ for $1 \leqslant j \leqslant q$.*

Assuming (3.6) for the moment, consider the case $p = 2$. Then we need to see that $(\psi^2 \mu/p^2 \mu) = (1 + \mu/4)^s$ lies in $1 + \widetilde{KO}(\mathbf{Q}P^n)$. This requires that 2^{2j} divide $\binom{s}{j}$ if j is even, and 2^{2j+1} divide $\binom{s}{j}$ if j is odd, for all $1 \leqslant j \leqslant n$. Applying (3.6) we obtain $\nu_2(s) - \nu_2(j) \geqslant 2j$ if j is even and $\nu_2(s) \geqslant 2j + 1$ if j is odd. Thus the expression for $\nu_2(B_n)$.

Now consider the case p odd. Then, by (3.3) we obtain that (3.5) is given as follows

$$(\psi^p \mu/p^2 \mu)^s = (\lambda + p^{-1} \mu^{(p-1)/2})^{2s}, \qquad \lambda \in 1 + \widetilde{KU}(\mathbf{Q}P^n).$$

Notice that $1 + x$ belongs to $1 + \widetilde{KU}(\mathbf{Q}P^n)$, if and only if $(1 + x)^2$ belongs to $1 + \widetilde{KO}(\mathbf{Q}P^n)$. Therefore it suffices to see that $(\lambda + p^{-1}\mu^{(p-1)/2})^s$ belongs to $1 + \widetilde{KU}(\mathbf{Q}P^n)$. This requires that p^j divide $\binom{s}{j}$ for all $1 \leqslant j \leqslant 2n/(p - 1)$. By

(3.6), we obtain $\nu_p\binom{s}{j} = \nu_p(s) - \nu_p(j) \geqslant j$, $1 \leqslant j \leqslant 2n/(p-1)$, and this gives the expression for $\nu_p(B_n)$.

PROOF OF (3.6). We prove it by induction on q. It is true for $q = 1$, since $\nu_p\binom{s}{1} \geqslant 1$ implies $s \equiv 0(p)$ and $\nu_p(1) = 0$, so $\nu_p\binom{s}{1} = \nu_p(s)$. Assume the result true for $q - 1$. Then we consider two cases.

Case 1. $q \not\equiv 1(p)$. We have

$$\binom{s}{q} = \frac{s(s-1)\cdots(s-q+2)}{1\cdots(q-1)}\frac{(s-q+1)}{q}$$

and

$$\nu_p\binom{s}{q-1} = \nu_p(s) - \nu_p(q-1) = \nu_p(s).$$

Now $(s - q + 1) \not\equiv 0 \ (p)$. Hence $\nu_p\binom{s}{q} = \nu_p(s) - \nu_p(q)$.

Case 2. $q \equiv 1(p)$. Then

$$\nu_p\binom{s}{q-1} = \nu_p(s) - \nu_p(q-1),$$

and

$$\nu_p\binom{s}{q} = \nu_p(s) - \nu_p(q-1) + \nu_p(s-q+1) - \nu_p(q).$$

But $\nu_p(s) \geqslant \nu(q-1) + q - 1$, by hypothesis, hence $\nu_p(s - q + 1) = \nu_p(q - 1)$, and we have $\nu_p\binom{s}{q} = \nu_p(s)$. This ends the proof of (3.6).

Note that K. Y. Lam in [12] used a similar number-theoretical argument to obtain the Atiyah-Todd number M_{n+1}, which was proven by Adams and Walker in [3] to be the J-order of the Hopf bundle over $\mathbb{C}P^n$. We offer here another proof of this fact.

DEFINITION 3.5. *Let $I(n)$ denote the ring of those power series $a_0 + a_1 z + \cdots + a_k z^k + \cdots$ in $Q[[z]]$ such that $a_i \in Z$ for $i \leqslant n$. Let $I_2(n)$ denote the subring of $I(n)$ for which $a_j \in 2Z$ if j is odd and $j \leqslant n$.*

PROPOSITION 3.6. *The J-order A_n of the Hopf bundle over $\mathbb{C}P^n$ is equal to the Atiyah-Todd number M_{n+1}.*

PROOF. Recall from [4] that the Atiyah-Todd number M_{n+1} was defined as the least number s such that the power series:

$$(3.7) \qquad L^s(z) = [\ln(1+z)/z]^s$$

belongs to $I(n)$.

Recall from Sanderson in [14], that

$$KO(\mathbb{C}P^n) = \begin{cases} Z[\alpha]/(\alpha^{[n/2]+1}) & \text{if } n \not\equiv 1 \ (4), \\ Z[\alpha]/(2\alpha^{[n/2]+1}, \alpha^{[n/2]+2}) & \text{if } n \equiv 1 \ (4), \end{cases}$$

$$KU(\mathbb{C}P^n) = Z[\beta]/(\beta^{n+1}),$$

and

$$H^*(\mathbb{C}P^n; Z) = Z[x]/(x^{n+1}).$$

The generators are $\alpha = r(\xi) - 2$, $\beta = \xi - 1$, $x = c_1(\xi)$, where as above ξ denotes the Hopf bundle over $\mathbb{C}P^n$. If c is the complexification,

$$c = \beta^2/(1 + \beta).$$

Also $ch\beta = e^x - 1$, so that $ch^{-1}(x) = \ln(1 + \beta)$.

Since 4ξ is KO-orientable over $\mathbb{C}P^n$ and $\alpha^2 \in KO(\mathbb{C}P_4^{n+4})$, is its Thom class, we have

$$ch(c\alpha^2) = x^4 \cdot bh(4\xi)$$

and

$$ch^{-1}(x^4) = c\alpha^2 \cdot cBh(-4\xi)$$

hence

$$(\ln(1 + \beta))^4 = \frac{\beta^4}{(1 + \beta)^2} cBh(-4\xi).$$

Therefore

$$cBh(\xi) = \frac{\beta}{\sqrt{1 + \beta} \ln(1 + \beta)}.$$

On the other hand $\rho^k(4\xi) = \psi^k(\alpha^2)/\alpha^2$. Now, in $\mathbb{C}P^{n+4}$, $\psi^k\alpha = k^2\alpha + \cdots + \alpha^{[n/2]+2} + \varepsilon\alpha^{(n+5)/2}$ with possibly $\varepsilon \neq 0$ when $n \equiv 1 \ (4)$. Therefore $\psi^k(\alpha^2) = k^4\alpha^2 + \cdots + a\alpha^{[n/2]+2}$ and $\rho^k(4\xi)$ as well as $\rho^k(4m\xi)$ have only terms $a_i\alpha^i$ for $i < (n + 1)/2$. Since this is a "canonical" torsion free summand of $KO(\mathbb{C}P^n)$, by the note after Corollary 2.5 it suffices that $Bh(4m\xi)$ be integral for $4m\xi$ to be J-trivial. Since $c: \alpha \to \beta^2 - \beta^3 + \cdots$ has leading coefficient 1 it indeed suffices that $cBh(4m\xi)$ be integral and since $1 + \beta$ is invertible in $1 + \widetilde{KU}(\mathbb{C}P^n)$ the claim follows.

REMARK. Consider the diagram:

$$\begin{array}{ccc} KO(QP^n) & \xrightarrow{\ c\ } & KU(QP^n) \\ \downarrow{\scriptstyle (\pi i)^*} & & \downarrow{\scriptstyle \pi^*} \\ & & KU(\mathbb{C}P^{2n+1}) \\ & & \downarrow{\scriptstyle i^*} \\ KO(\mathbb{C}P^{2n}) & \xrightarrow{\ c\ } & KU(\mathbb{C}P^{2n}) \end{array}$$

Here c and $(\pi i)^*$ are monomorphisms. Moreover

$$c(KO(\mathbf{C}P^{2n})) = (\pi i)^* KU(\mathbf{Q}P^n)$$

thus $\varphi: KU(\mathbf{Q}P^n) \to KO(\mathbf{C}P^{2n})$ given by $\varphi(\mu) = \alpha$ is an isomorphism of ψ-rings and we could have computed B_n from our knowledge of A_{2n}. Namely, we have: $\nu_p(B_n) = \nu_p(A_{2n})$ for p odd. This follows from the fact that $\varphi Bh(\xi_Q)$ $= Bh(2\xi_c) = L(\alpha)$ and that from (3.6) and (3.7) we have

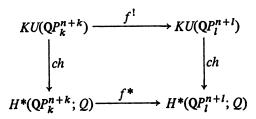$$2k\xi_c \text{ is } J\text{-trivial if and only if } L^k(x) \in I(n)$$

and

$$k\xi_Q \text{ is } J\text{-trivial if and only if } L^k(x) \in I_2(n).$$

To complete the computation of B_n we would only need to use

$$\psi^2(\mu) = 4\mu + \mu^2.$$

4. Homotopy equivalences and their representations in ordinary cohomology and K-theory. The object of this section is to determine the homomorphisms $f^!: KU(\mathbf{Q}P_k^{n+k}) \to KU(\mathbf{Q}P_l^{n+l})$ induced by the possible stable homotopy equivalences between the two spaces. This will provide the vital necessary conditions in Theorem 1.2. The main tool here is the commutativity of the diagram

$$
\begin{array}{ccc}
KU(QP_k^{n+k}) & \xrightarrow{\ f^!\ } & KU(QP_l^{n+l}) \\
\Big\downarrow{ch} & & \Big\downarrow{ch} \\
H^*(QP_k^{n+k}; Q) & \xrightarrow{\ f^*\ } & H^*(QP_l^{n+l}; Q)
\end{array}
$$

for a stable homotopy equivalence $f: \mathbf{Q}P_l^{n+l} \to \mathbf{Q}P_k^{n+k}$. Note that f^* and $f^!$ are taken in conjunction with suspension isomorphism and Bott periodicity. We shall also use the fact that $f^!$ must preserve both $KO(\mathbf{Q}P_r')$ and $K\,\mathrm{Sp}(\mathbf{Q}P_r')$ embedded in $KU(\mathbf{Q}P_r')$. This fact implies that $k \equiv l\ (2)$ for a possible equivalence. Here the groups are as defined in §3 and we will continue to use the notation there introduced.

First observe that f^* can be thought of as a diagonal $(n+1) \times (n+1)$ matrix $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_i = \pm 1$. Since each of the spaces in question admits a self-equivalence which induces the matrix $(-1, -1, \ldots, -1)$ in cohomology, the matrices (ε_i) and $(-\varepsilon_i)$ can be thought of as equivalent. Since ch is a rational isomorphism, $f^!: KU(\mathbf{Q}P_k^{n+k}) \otimes Q \to KU(\mathbf{Q}P_l^{n+l})$

$\otimes$ Q is uniquely determined by f^*; namely $f^! = ch^{-1} f^* ch$. Thus $f^!$ is, in some sense, an adjoint of f^*. Since $f^!$ must be integral we obtain restrictions on the possible homomorphisms f^*.

Let us compute $f^!$ on μ^k when f^* is the identity matrix I. We have $f^!(\mu^k) = ch^{-1} I ch \mu^k$. Recall that $\mu^k = U_{k\xi}^U$, hence $ch\mu^k = U_{k\xi}^H bh_c(k\xi)$. Now f^* being the "identity" replaces $U_{k\xi}^H$ by $U_{l\xi}^H$ and so

$$f^!(\mu^k) = ch^{-1}(U_{l\xi}^H bh(k\xi)).$$

We have, by (2.5) and (2.6),

$$f^!(\mu^k) = U_{l\xi}^U \cdot Bh_c(-l\xi) \cdot Bh_c(k\xi) = \mu^l \cdot Bh_c((k-l)\xi).$$

From KU considerations alone, we conclude that $Bh_c((k-l)\xi)$ must be integral.

We now determine the class $Bh_c(\xi)$ as a power series. Recall that $ch\mu = e^x + e^{-x} - 2$, with $y = x^2$. Thus

$$ch\mu = y + 2y^2/4! + 2y^3/6! + \cdots .$$

Thus $ch\mu$ is a power series which begins with y and hence it can be inverted (in the sense of substitution of series) to obtain

$$ch^{-1}y = \mu + a_2\mu^2 + \cdots \quad \text{and} \quad Bh(-\xi) = 1 + \sum a_i \mu^i.$$

We will denote the power series $1 + \sum_{r>0} a_r x^r$ by $t(X)$. It is elementary to determine the coefficients a_r, which we include here for the sake of completeness:

$$(4.1) \qquad a_r^{-1} = (-1)^r (r+1)(2r+1)\binom{2r}{r}.$$

Now $f^!$ must preserve both KO and $K\mathrm{Sp}$. Moreover $KF(\mathbf{Q}P_k^{n+k}) \subset KU(\mathbf{Q}P_k^{n+k})$ is generated by $\mu^k, 2\mu^{k+1}, \mu^{k+2}, \ldots$ if $F = \mathrm{Sp}$ and k is odd or if $F = O$ and k is even. It follows that $Bh_c((k-l)\xi) = (T(\mu))^{k-l}$ not only belongs to $I(n)$ but also to $I_2(n)$.

We obtain:

PROPOSITION 4.1. *There exists a stable homotopy equivalence* $f: \mathbf{Q}P_l^{n+l} \to \mathbf{Q}P_k^{n+k}$ *with* $f^* = (1, 1, \ldots, 1)$ *if and only if* $k = l(B_n)$. *In this case* $f^!\mu^{k+i} = \mu^{l+i} \cdot (T(\mu))^{l-k}$.

Note: In [11], Held and Sutter have proved general results which include (4.1), as a special case.

Our next aim is to prove that, save multiplication by $-I$, the only possible matrices which can occur as f^* are $(1, 1, \ldots, 1)$ and $(-1, 1, 1, \ldots, 1)$. Since an

equivalence f can always be made cellular, it suffices to prove our assertion for 3×3 matrices, i.e. for possible equivalences between Thom spaces over $\mathbf{Q}P^2$. To this end we first prove:

LEMMA 4.2. *Let $f: \mathbf{Q}P_l^{l+1} \to \mathbf{Q}P_k^{k+1}$ be a stable homotopy equivalence such that $f^* = (-1, 1)$. Then $k + l \equiv 0$ (24).*

PROOF. As before, we compute $f^!(\mu^k)$. We have

$$f^! \mu^k = ch^{-1} f^* ch \mu^k = ch^{-1} f^* (U_{k\xi}^H bh(k\xi)).$$

But $f^* = (-1, 1)$, hence

$$\begin{aligned} f^! \mu^k &= ch^{-1}(U_{l\xi}^H(bh(k\xi) - 2)) \\ &= -U_{l\xi}^U \cdot Bh(-l\xi)(2 - Bh(k\xi)) \\ &= -U_{l\xi}^U \cdot (Bh((k - l)\xi) - 2Bh(-l\xi)) \\ &= -\mu^l \cdot (Bh(k - l)\xi)(2Bh(-k\xi) - 1). \end{aligned}$$

Hence

$$f^! \mu^k = -\mu^l \cdot T^{l-k}(\mu)(2T^k(\mu) - 1)$$

where we have set $T^r(\mu) = (T(\mu))^r$. Our requirement now is

(4.3) $$T^{l-k}(x)(2T^k(x) - 1) \in I_2(1).$$

But $T(x) = 1 - x/12 + x^2/90 + \cdots$, we have

$$\left(1 - \frac{x}{12}\right)^{l-k}\left(2\left(1 - \frac{x}{12}\right)^k - 1\right) = 1 - \frac{l+k}{12}x + \cdots \in I_2(1)$$

and so $l + k \equiv 0$ (24) as was claimed.

REMARK. The formula (4.3) holds generally for equivalences f such that $f^* = (-1, 1, 1, \dots, 1)$.

LEMMA 4.4. *Let $f: \mathbf{Q}P_l^{l+2} \to \mathbf{Q}P_k^{k+2}$ be a stable homotopy equivalence. Then, up to multiplication by $-I$, either $f^* = (1, 1, 1)$ or $f^* = (-1, 1, 1)$.*

PROOF. It suffices to rule out the possibilities $(1, -1, 1)$ and $(1, 1, -1)$ for f^*. We assume that f is cellular. By restriction to $f_1: \mathbf{Q}P_l^{l+1} \to \mathbf{Q}P_k^{k+1}$ we find, in the case $(1, -1, 1)$, that $k + l \equiv 0$ (24). On the other hand, we also have an equivalence on the coskeleta: $f_2: \mathbf{Q}P_{l+1}^{l+2} \to \mathbf{Q}P_{k+1}^{k+2}$ with $f_2^* = (-1, 1)$ and so $k + l + 2 \equiv 0$ (24) which contradicts the first congruence. This rules out the case $f^* = (1, -1, 1)$. For the case $f^* = (1, 1, -1)$ we compute $f^!(\mu^k)$. We have

$$f^* ch \mu^k = f^* [y^k bh(k\xi)] = f^* \left[y^k \left(1 + \frac{2y}{4!} + \frac{2y^2}{6!} \right)^k \right]$$

$$= f^* \left[y^k \left(1 + \frac{2k}{4!} y + \frac{k(5k-1)}{1440} y^2 \right) \right]$$

$$= y^l \left[bh(k\xi) - \frac{k(5k-1)}{720} y^2 \right].$$

Here $y^l = U_{l\xi}^H$ and since $ch^{-1} y^{l+2} = \mu^{l+2} = U_{l\xi}^U \cdot \mu^2$ we obtain

$$f^l \mu^k = U_{l\xi}^U \cdot Bh(-l\xi) \left[Bh(k\xi) - \frac{k(5k-1)}{720} \mu^2 \right]$$

$$= U_{l\xi}^U \cdot Bh((k-l)\xi) \left[1 - Bh(-k\xi) \frac{k(5k-1)}{720} \mu^2 \right]$$

$$= \mu^l \cdot Bh((k-l)\xi) \left[1 - \frac{k(5k-1)}{720} \mu^2 \right].$$

This last equality holds because we are in $KU(\mathbf{Q}P_l^{l+2})$. Our integrality condition imposes then

$$\left(1 - \frac{k(5k-1)}{720} x^2 \right) \left(1 - \frac{x}{12} + \frac{x^2}{90} \right)^{l-k} \in I_2(2).$$

This yields $l - k \equiv 0$ (24), since the coefficient of x is $(l - k)/12$. One finds the coefficient of x^2 to be

$$\binom{l-k}{2} \frac{1}{12^2} + \frac{l-k}{90} - \frac{k(5k-1)}{720}$$

and since this coefficient must be an integer we obtain the condition

$$5(l-k)(l-k-1) + 2^4(l-k) - 2k(5k-1) \equiv 0 \ (24 \times 60).$$

Calculating modulo 24, since $l - k \equiv 0$ (24), we obtain

(4.5) $k(5k-1) \equiv 0$ (12).

On the other hand the map on the coskeleta

$$f_2 : \mathbf{Q}P_{l+1}^{l+2} \to \mathbf{Q}P_{k+1}^{k+2}$$

induced by f yields $f_2^* = (1, -1)$ which, by Lemma 4.2, implies $k + l + 2 \equiv 0(24)$. This together with $k - l \equiv 0(24)$ yields

$$k \equiv -1(12) \quad \text{and} \quad k(5k-1) \equiv 6(12).$$

This, however, is in contradiction with the congruence (4.5).

COROLLARY 4.6. *Let f: $\mathbf{Q}P_l^{n+l} \to \mathbf{Q}P_k^{n+k}$ be a stable homotopy equivalence. Then, up to multiplication by $-I$, either $f^* = (1, 1, \ldots, 1)$ or $f^* = (-1, 1, \ldots, 1)$.*

We now proceed to study $f^!$ when f^* is of the form $(-1, 1, \ldots, 1)$. By (4.3) we have

$$T^{l-k}(x)[2T^k(x) - 1] \in I_2(n).$$

Moreover, by considering the equivalence f_2: $\mathbf{Q}P_{l+1}^{n+l} \to \mathbf{Q}P_{k+1}^{n+k}$ induced by f we have, by Proposition 4.1, $l - k \equiv O(B_{n-1})$. We are now ready to prove

PROPOSITION 4.7. *If f: $\mathbf{Q}P_l^{n+l} \to \mathbf{Q}P_k^{n+k}$ is a stable homotopy equivalence with $f^* = (-1, 1, \ldots, 1)$ then $k - l \equiv O(B_{n-1})$ and $k + l \equiv O(B_n)$. Moreover*

$$f^!(\mu^k) = \mu^l T^{l-k}(\mu)[2T^k(\mu) - 1]$$

and

$$f^!(\mu^{k+i}) = \mu^{l+i} T^{l-k}(\mu)$$

for $i > 0$.

This proposition provides the desired necessary condition in Theorem 1.2. We first prove two lemmas. The first is a simple statement about the growth of B_n.

LEMMA 4.8. *If $2k \equiv O(B_{n-1})$ and $n > 2$ then $k \equiv O(B_{[n/2]})$.*

This is a very weak statement which follows upon direct inspection of the formula for $\nu_2(B_n)$ and the inequality $\log_2 n < n/2$.

LEMMA 4.9. *If $2T^k(x) - 1 \in I_2(n)$ then $T^{2k}(x) \in I_2(n)$.*

PROOF. For $n = 1$ the claim is obvious. Suppose that

$$2T^k(x) - 1 \in I_2(n) \subset I_2(n - 1).$$

We have, by induction, $T^{2k}(x) \in I_2(n - 1)$ and by Lemma (4.8)

$$T^k(x) \in I_2([n/2])$$

i.e.

$$T^k(x) = 1 + a_1 x + \cdots + a_{[n/2]} x^{[n/2]} + \tfrac{1}{2}(a_{[n/2]} x^{[n/2]+1} + \cdots + a_n x^n) + \cdots$$

where a_i are integers which are even for i odd, $i \leqslant n$.

Since $2 + 2[n/2] > n$ we conclude that

$$[T^k(x)]^2 = T^{2k}(x) \in I_2(n).$$

PROOF OF PROPOSITION (4.7). For $n = 1$ the statement is that of Lemma 4.2. Note that the formulas in Proposition (4.7) follow from (4.3) and Proposition (4.1) by collapsing the bottom cells.

It remains to prove that the condition

$$T^{l-k}(x)[2T^k(x) - 1] \in I_2(n)$$

together with $l - k \equiv O(B_{n-1})$ implies $l + k \equiv O(B_n)$. Since $l - k \equiv O(B_{n-1})$ we have $T^{l-k} \in I_2(n - 1)$ and hence

$$2T^k(x) - 1 \in I_2(n - 1).$$

By Lemma 4.8 this implies that $T^{2k}(x) \in I_2(n - 1)$ and, by Lemma (4.8), $T^k(x) \in I_2([n/2])$ as long as $n > 2$. We can write

$$T^k(x) = 1 + a_1 x + \cdots + a_{[n/2]} x^{[n/2]}$$
$$+ \tfrac{1}{2}(a_{[n/2]+1} x^{[n/2]+1} + \cdots + a_{n-1} x^{n-1} + \alpha x^n + \cdots)$$

and

$$T^{l-k}(x) = 1 + b_1 x + \cdots + b_{n-1} x^{n-1} + \beta x^n + \cdots$$

where the a_i's and b_i's are integers which are even for i odd and α, β are some rational numbers.

We have

$$2T^k(x) - 1 = 1 + \cdots + a_{n-1} x^{n-1} + 2\alpha x^n + \cdots$$

and

$$T^{2k}(x) = 1 + \cdots + (\theta + 2\alpha) x^n + \cdots$$

where θ is an integer which is even if n is odd. This follows since

$$2 + 2[n/2] > n.$$

Since

$$T^{l-k}(x)[2T^k(x) - 1] \in I_2(n)$$

$2\alpha + \beta$ must be an integer of appropriate parity. This, clearly, implies that $T^{l-k}(x) \cdot T^{2k}(x) = T^{l+k}(x) \in I_2(n)$ i.e. $l + k \equiv O(B_n)$.

It remains to verify the case $n = 2$. This is a routine calculation with $T(x) = 1 - x/12 + x^2/90 + \cdots$.

We have

$$T^{l-k}(x)[2T^k(x) - 1]$$

$$= \left(1 - \frac{l-k}{12}x + \left[\binom{l-k}{2}\frac{1}{12^2} + \frac{l-k}{90}\right]x^2\right)$$

$$\cdot \left(1 - \frac{2k}{12}x + \left[\binom{k}{2}\frac{2}{12^2} + \frac{2k}{90}\right]x^2\right)$$

$$= 1 - \frac{l+k}{12}x + \left[\frac{2k(l-k)}{12^2} + \binom{l-k}{2}\frac{1}{12^2} + \binom{k}{2}\frac{2}{12^2} + \frac{l+k}{90}\right]x^2$$

and so, by integrality of the second coefficient:

$$20k(l - k) + 5(l - k)(l - k - 1) + 10k(k - 1) + 16(l + k)$$

$$\equiv O(2^5 \cdot 2^2 \cdot 5).$$

Using $l - k \equiv O(2^3 \cdot 3)$ and $B_1 = 2^3 \cdot 3$, $B_2 = 2^5 \cdot 3^2 \cdot 5$ one easily obtains

$$l + k \equiv O(2^5); \quad (l + k) \equiv O(3^2); \quad (l + k) \equiv O(5).$$

5. Construction of the homotopy equivalences. To construct a stable homotopy equivalence between $Q P_k^{n+k}$ and $Q P_l^{n+l}$, when the arithmetical conditions permit it, we remove some of the top cells and study the attaching maps. More precisely, consider the cofibration

$$Q P_k^{k+t-1} \to Q P_k^{n+k} \to Q P_{t+k}^{n+k}.$$

Whenever $2(t + k) > n + k$, this cofibration desuspends to give

$$\Sigma^{-1} Q P_{t+k}^{n+k} \to Q P_k^{k+t-1} \to Q P_k^{n+k}.$$

Recall, that given two cofibrations and a commutative diagram

$$
\begin{array}{ccccc}
A & \longrightarrow & B & \longrightarrow & C \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
A' & \longrightarrow & B' & \longrightarrow & C'
\end{array}
$$

with f and g homotopy equivalences, there exists a homotopy equivalence $h: C \to C'$.

With this in mind, we shall examine the cofibrations which produce the spaces $Q P_k^{n+k}$.

First of all we shall introduce standard equivalences

$$Q P_{r+B_n}^{n+r+B_n} \to \Sigma^{4B_n} Q P_r^{n+r}.$$

This amounts to fixing a J-trivialization of $B_n\xi$ over $\mathbf{Q}P^n$.

Consider now the sequence

$$\Sigma^{-1}QP^{n+k}_{t+k} \longrightarrow QP^{k+t-1}_{k} \longrightarrow QP^{n+k}_{k} \longrightarrow QP^{n+k}_{t+k}$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$\Sigma^{4k-1}QP^n_t \longrightarrow (QP^{t-1})^{k\xi} \longrightarrow (QP^n)^{k\xi} \longrightarrow \Sigma^{4k}QP^n_t$$

The last vertical arrow is the "standard" equivalence as long as we make sure that $k \equiv O(B_{n-t})$.

We now restrict our attention to the space $(\mathbf{Q}P^{t-1})^{k\xi}$. If $k \equiv O(B_{t-1})$ then $k\xi$ is J-trivial and $(\mathbf{Q}P^{t-1})^{k\xi}$ has the same homotopy type as $S^{4k} \vee \Sigma^{4k}\mathbf{Q}P^{t-1}$. It is our aim now to construct specific homotopy classes of retractions

$$(\mathbf{Q}P^{t-1})^{k\xi} \to S^{4k}.$$

Let G_m denote the space of self-homotopy equivalences of S^{m-1}. Then G_m e.g. [14], is a monoid under composition with classifying space BG_m. This space classifies $(m-1)$-spherical fibrations.

The cofibration

$$\mathbf{Q}P^{t-1} \xrightarrow{i} \mathbf{Q}P^n \xrightarrow{\pi} \mathbf{Q}P^n_t$$

gives rise to an exact sequence of groups:

$$[\mathbf{Q}P^n_t, BG_{4k}] \xrightarrow{\pi^*} [\mathbf{Q}P^n, BG_{4k}] \xrightarrow{i^*} [\mathbf{Q}P^{t-1}BG_{4k}].$$

We shall use the symbol $[k\xi]$ for the element in $[\mathbf{Q}P^n, BG_{4k}]$ which corresponds to the spherical fibration associated with $k\xi$. If $k \equiv O(B_{t-1})$ then $i^*[k\xi] = 0$ and so there exists an element $\alpha \in [\mathbf{Q}P^n_t, BG_{4k}]$ such that $\pi^*\alpha = [k\xi]$.

Note that π induces a map between Thom spaces

$$(\mathbf{Q}P^n)^{k\xi} \xrightarrow{\tilde{\pi}} (\mathbf{Q}P^n_t)^\alpha$$

which we shall consider cellular. Since $(\mathbf{Q}P^n_t)^\alpha$ has no cells of dimension r, $4k < r < 4k + 4t(4k + 4t)$, the restriction of this map to $(\mathbf{Q}P^{t-1})^{k\xi} \subset (\mathbf{Q}P^n)^{k\xi}$ yields a map

$$(\mathbf{Q}P^{t-1})^{kH} \xrightarrow{g_\alpha} S^{4k}$$

which is clearly a retraction. The homotopy class of g_α depends only on the choice of α.

Let us denote by $F_{m-1} \subset G_m$ the submonoid of those self-equivalences of the $(m-1)$-sphere which preserve the base point. Using the suspension map we have an inclusion $G_n \subset F_n$ and this is a $(2n-5)$-equivalence (see e.g. [10]). We have $F_n = \Omega^n S^n$. Suppose $X = \sum Y$ then, if $m > 2 \dim X$, we have

$$[X, BG_m] \cong [\textstyle\sum Y, EG_m] \cong [Y, G_m] \cong [Y, F_m]$$

$$\cong [Y, \Omega^m S^m] \cong [\textstyle\sum^m Y, S^m] \cong [\textstyle\sum^{m-1} X, S^m].$$

Let $\theta \colon [X, BG_m] \cong [\sum^{m-1} X, S^m]$. We recall the following result which appears in Wall [17, Proposition 3.7]:

PROPOSITION 5.1. *Let α be an $(m-1)$-spherical fibration over $\sum Y$, where $m > 2 \dim \sum Y$. Then*
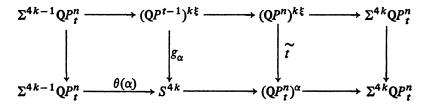
$$\textstyle\sum^m Y \xrightarrow{\theta(\alpha)} S^m \to (\sum Y)^\alpha$$

is a cofibration.

If we assume that $2t > n$ so that $\mathbf{Q}P_t^n$ is a suspension then we get the cofibration

$$\textstyle\sum^{4k-1} \mathbf{Q}P_t^n \xrightarrow{\theta(\alpha)} S^{4k} \to (\mathbf{Q}P_t^n)^\alpha.$$

The following commutative diagram relates $\theta(\alpha)$ and g_α:

The inclusion $BG_{4k} \subset BG_{4k+1}$ induces isomorphisms

$$[X, BG_{4k}] \to [X, BG_{4k+1}]$$

if the dimension of X is small compared to k. This way we can consider $\alpha \in [\mathbf{Q}P_t^n, BG_{4k}]$ to be an element of $[\mathbf{Q}P_t^n, BG_{4k+r}]$. All resulting maps would be suspended r times.

LEMMA 5.2. *Let $k\xi$ and $l\xi$ be multiples of the Hopf bundle over $\mathbf{Q}P^n$. If $k\xi$ as well as $l\xi$ are J-trivial over $\mathbf{Q}P^{t-1}$ and $(k+l)\xi$ is J-trivial over $\mathbf{Q}P^n$ then there exist $\alpha, \beta \in [\mathbf{Q}P_t^n, BG_{4(k+l)}]$ such that $\pi^* \alpha = [k\xi] + 4l$, $\pi^* \beta = [l\xi] + 4k$ and $\alpha + \beta = 0$.*

PROOF. The cofibration

$$\mathbf{Q}P^{t-1} \xrightarrow{i} \mathbf{Q}P^n \xrightarrow{\pi} \mathbf{Q}P_t^n$$

yields an exact sequence:

$$[\Sigma\, \mathbf{Q}P^{t-1}, BG_r] \xrightarrow{\delta} [\mathbf{Q}P_t^n, BG_r] \xrightarrow{\pi^*} [\mathbf{Q}P^n, BG_r] \xrightarrow{i^*} [\mathbf{Q}P^{t-1}, BG_r]$$

where $r = 4(k + l)$.

Since $k\xi$ and $l\xi$ are J-trivial over $\mathbf{Q}P^{t-1}$ there exist elements $\alpha', \beta' \in [\mathbf{Q}P_t^n, BG_{4(k+l)}]$ with

$$\pi^*\alpha' = [k\xi] + 4l \quad \text{and} \quad \pi^*\beta' = [l\xi] + 4k.$$

Since $\pi^*(\alpha' + \beta') = 0$ there exists an element $\gamma \in [\Sigma\, \mathbf{Q}P^{t-1}, BG_{4(k+l)}]$ such that $\delta\gamma = \alpha' + \beta'$. Let $\alpha = \alpha'$ and $\beta = \beta' - \delta\gamma$.

We now are ready to construct the desired homotopy equivalences.

PROPOSITION 5.3. *If $(k - l)\xi$ is J-trivial over $\mathbf{Q}P^{n-1}$ and $(k + l)\xi$ is J-trivial over $\mathbf{Q}P^n$ with $n > 2$ then $\mathbf{Q}P_k^{n+k}$ and $\mathbf{Q}P_l^{n+l}$ are of the same stable homotopy type.*

PROOF. From $k - l \equiv O(B_{n-1})$ and $k + l \equiv O(B_n)$ we obtain $2k \equiv 2l \equiv O(B_{n-1})$ and $k \equiv l \equiv O(B_{[n/2]})$ for $n - 1 > 1$. Choosing $t - 1 = [n/2]$ all arguments in (5.2) go through. Let $\alpha, \beta \in [\mathbf{Q}P_t^n, BG_{4(k+l)}]$ be the pullbacks of $[k\xi]$ and $[l\xi]$ respectively and let $\alpha + \beta = 0$.

Note that $\mathbf{Q}P_{k+1}^{t-1+k} = \mathbf{Q}P_k^{t-1+k}/S^{4k}$ is canonically equivalent with $\Sigma^{4k}\mathbf{Q}P^{t-1}$. Using the collapsing map $\mathbf{Q}P_k^{t-1+k} \xrightarrow{p_k} \mathbf{Q}P_{k+1}^{t-1+k}$ together with g_α we get a homotopy equivalence

$$(\mathbf{Q}P^{t-1})^{kH} \xrightarrow{g_\alpha \vee p_k} S^{4k} \vee \Sigma^{4k}\mathbf{Q}P^{t-1}.$$

Now consider the diagram of cofibrations and maps between them:

The map φ is obtained using the canonical equivalences $(\mathbf{Q}P^{t-1})^{B_{t-1}\xi}$ $\sim \Sigma^{4B_{t-1}}(\mathbf{Q}P^{t-1} \vee S^0)$. Since $\alpha + \beta = 0$, and so $-\theta(\alpha) = \theta(\beta)$, all rectangles are commutative and all spaces on the right are of the same homotopy type.

With this proposition we complete the proof of Theorem 1.2 for $n > 2$. The same argument yields the homotopy equivalences needed for Theorem 1.1.

Let $\mathcal{E}^s(X)$ denote the group of homotopy classes of stable self equivalences of the space X. There is a natural representation

$$\mathcal{E}^s(X) \xrightarrow{e} \mathrm{Aut}\,(H^*(X; Z))$$

where $H^*(X; Z)$ is considered to be an abelian group only.

For stunted complex projective spaces we have

THEOREM 5.4. *The image of e for $X = \mathbf{C}P_k^{n+k}$ for $n > 4$ is generated by*
(i) $-I$ and $(-1, 1, 1, \ldots, 1)$ if $2k \equiv O(A_n)$,
(ii) $-I$ and $(1, -1, 1, \ldots, 1)$ if $k + 1 \equiv O(A_{n-1})$,
(iii) $-I$ and $(1, \ldots, 1, -1, 1)$ if $k + n + 2 \equiv O(A_{n-1})$,
(iv) $-I$ and $(1, \ldots, 1, 1, -1)$ if $2(k + n + 1) \equiv O(A_n)$, *and by $-I$ alone otherwise.*

For stunted quaternionic projective spaces we have, similarly,

THEOREM 5.5. *The image of e for $X = \mathbf{Q}P_k^{n+k}$ is generated by $-I$ and $(-1, 1, 1, \ldots, 1)$ if $2k \equiv O(B_n)$ and by $-I$ alone otherwise.*

These theorems follow directly from the constructions of homotopy equivalences and the necessary conditions obtained via K-theory. Only (ii) or the dual statement (iii) of Theorem 1.1 needs a separate proof. We shall prove (ii). To this end we need only construct a self-equivalence of the form $(1, -1, 1, \ldots, 1)$. Consider the cofibration

$$\Sigma^{-1}\mathbf{C}P_{k+1}^{n+k} \to S^{2k} \to \mathbf{C}P_k^{n+k}.$$

Since $k + 1 \equiv O(A_{n-1})$ we have $\mathbf{C}P_{k+1}^{n+k} \sim S^{2k+2} \vee \mathbf{C}P_{k+2}^{n+k}$ and the diagram

$$
\begin{array}{ccc}
S^{2k+1} \vee \Sigma^{-1}\mathbf{C}P_{k+2}^{n+k} & \longrightarrow & S^{2k} \\
{\scriptstyle -1 \vee 1}\downarrow & & \downarrow{\scriptstyle 1} \\
S^{2k+1} \vee \Sigma^{-1}\mathbf{C}P_{k+2}^{n+k} & \longrightarrow & S^{2k}
\end{array}
$$

is homotopy commutative since $2\pi_{2k+1}(S^{2k}) = 0$. This yields the desired self-equivalence.

REMARK. This, in principle, solves the classification problem of Thom spaces over stunted complex and quaternionic projective spaces using the results of [8].

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